Amenable subgroups of Homeo($\mathbb{R}$) with large characterizing quotients

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In [Be], the author claims a certain classification result for subgroups of $\mathrm{Homeo}_+(\mathbb{R})$ - the group of orientation preserving homeomorphisms of the line. Shortly after the appearance of the first version of his paper, a well known immediate counterexample was pointed out (by us, Matthew Brin, and Andrés Navas). The author has stated he can save the paper by essentially adding a hypothesis about the existence of a freely acting element.

In this paper we disprove a major claim of the last version of [Be] (Theorem B*). For the sake of completeness let us quote the statement of this theorem from [Be]:

Theorem B. Let G be a subgroup of $Homeo_+(\mathbb{R})$ with a freely acting element. Then either the quotient group G/H_G is not amenable or the quotient group is solvable with solvability length not greater than 2. Specified alternative is strict and so it does not allow the simultaneous fulfillment of the conditions.*

We prove the following theorem to contradict this statement.

Theorem 1. There exists a finitely generated solvable subgroup Γ of $\mathrm{Homeo}_+(\mathbb{R})$ with a freely acting element such that Γ/H_Γ has solvability length greater than 2.

The subgroup H_Γ is defined in [Be]. For the sake of completeness we will recall the definition of it below but the only thing the reader of this paper needs to know (about the definition of H_Γ) is that a freely acting element of Γ does not belong to H_Γ. The quotient Γ/H_Γ turns out to be a very meaningful object. It has some characterizing power, therefore Theorem B* seemed very interesting to us.

Besides the quotient Γ/H_Γ, another major characteristics is the notion of *minimal set*. Given a subgroup $\Gamma \leq \mathrm{Homeo}_+(\mathbb{R})$, a non-empty closed Γ-invariant subset $E \subseteq \mathbb{R}$ is called a minimal set of Γ if it does not contain a proper non-empty closed Γ-invariant subset. If there is no such set E then by definition we assume that the minimal set is empty.

For finitely generated subgroups of $\mathrm{Homeo}_+(\mathbb{R})$, there exists a non-empty minimal set. (cf.[Be] or [N]).

Let us now *quote* the following definition from [Be].

Definition 1. For a subgroup Γ of $\mathrm{Homeo}_+(\mathbb{R})$, the normal subgroup H_Γ is defined as follows:

1) if the minimal set (denoted by $E(\Gamma)$) is neither empty nor discrete then

$$H_\Gamma = \{h \in \Gamma \mid E(\Gamma) \subseteq Fix(h)\}$$

2) if the minimal set is non-empty and discrete then $H_\Gamma = \Gamma^s$ (here $\Gamma^s = \cup_{t\in\mathbb{R}} St_\Gamma(t)$, i.e. Γ^s denotes the union of stabilizers of all points $t \in \mathbb{R}$).

3) if the minimal set is empty then $H_\Gamma = 1$.

The reader is referred to [Be] for well definedness of the subgroup H_Γ. Notice that the set Γ^s is not necessarily a subgroup of Γ, in general. However, it is a very nice lemma [Be] that a subgroup generated by Γ^s either coincides with Γ^s or coincides with Γ itself.

Remark 1. We can disprove at least one more major statement of [Be], namely, Theorem B, by a completely different but equally short construction. The statement of this theorem involves the notion of projective measures, and we needed to use some results from [Be] about (groups admitting) projective measures to disprove this claim. (so our construction/argument would not be self-contained).

Remark 2. The condition about existence of a freely acting element is indeed very interesting. For example if all non-identity elements of a subgroup Γ of $\mathrm{Homeo}_+(\mathbb{R})$ act freely then the group is Archimedean with a bi-invariant order, and therefore (by Hölder's Theorem) it is Abelian ([cf.N]). If every non-identity element has at most one fixed point then the group is meataabelian, even more specifically, it is a subgroup of the affine group $\mathit{Aff}(\mathbb{R})$ as proved by Barbot [Ba] and Kovacevic [K] (see [FF] for the history of this result). If every non-identity element has at most N fixed points, where N is a fixed positive integer, then we do not know what are the algebraic implications of this condition but it seems to us that this is an enormous restriction on the group. For example, if Γ contains two distinct elements a, b such that $a^m = b^m$ for some integer m (for example, the Klein bottle group $K = \langle a, b \mid a^2 = b^2 \rangle$), then such Γ cannot satisfy the above condition for any fixed N - the element ab^{-1} necessarily has infinitely many fixed points.

Note: We are thankful to A.Navas for his interest in this work.

PROOF OF THEOREM 1.

We intend the construct a finitely generated solvable subgroup Γ of $\mathrm{Homeo}_+(\mathbb{R})$ such that Γ contains a freely acting element and Γ/H_Γ is not metaabelian. The only thing we need to know about the definition H_Γ is that a freely acting element does not belong to it.

Let Γ be a group generated by three elements $t, a, b \in \Gamma$. Let us assume that the following conditions hold:

(i) Γ is solvable;

(ii) Γ is left-orderable with a left order $<$, moreover, $1 < t < a < b$;

(iii) $tat^{-1} = a^2$;

(iv) $aba^{-1} = b^2$;

To state the last four conditions we need to introduce some notations: let C denotes the cyclic subgroup of Γ generated by t, G denotes the subgroup generated by t and a, and $S = \{a^{-i}ba^i \mid i \in \mathbb{Z}\}\}$.

(v) $[t^ibt^{-i}, a^jba^{-j}] = 1$, for all $i, j \in \mathbb{Z}$.

(vi) if $g \in C, f \in \Gamma \backslash C, 1 < f$ then $g < f$;

(vii) if $g \in G, f \in \Gamma \backslash G, 1 < f$ then $g < f$.

(viii) there exists $h \in S$ such that $g < h < t^nbt^{-n}$ for all $g \in G, n \in \mathbb{Z}$.

We are postponing the construction of Γ with properties (i)-(viii) till the end.

Let us now observe some implications of conditions (i)-(viii):

First, let us observe that $b \in \Gamma^{(2)}$ [$\Gamma^{(n)}$ denotes the n-th commutator subgroup of Γ, for all $n = 1, 2, \ldots$]. Indeed, by (iii), $a = [t, a] \in \Gamma^{(1)}$, and by (iv), $b = [a, b] \in \Gamma^{(1)}$. Then, again by (iv), $b \in \Gamma^{(2)}$. Since $b \neq 1$, we conclude that Γ is not metaabelian.

Because of (ii), Γ is embeddable in $\mathrm{Homeo}_+(\mathbb{R})$. Moreover, we can embed Γ faithfully in $\mathrm{Homeo}_+(\mathbb{R})$ such that the following conditions hold:

(c1) if $g_1, g_2 \in \Gamma, g_1 < g_2$ then $g_1(0) < g_2(0)$ (in particular, $g(0) > 0$ for all positive $g \in \Gamma$);

(c2) there exists $h \in S$ such that $g(0) < h(0) < t^nbt^{-n}(0)$ for all $g \in G, n \in \mathbb{Z}$;

(c3) Γ has no fixed point.

We intend to show that if all conditions (i)-(viii) and (c1)-(c3) hold then the element b necessarily acts freely.

Before proving this let us observe that it is sufficient for our purpose: let $\bar{\Gamma} = \Gamma/H_\Gamma$, and $\bar{b}$ be the image of b in $\bar{\Gamma}$ under the quotient.

Since $b \in \Gamma^{(2)}$ and $b \notin H_\Gamma$ (because b acts freely) we obtain that $\bar{b} \in \bar{\Gamma}^{(2)}$ and $\bar{b} \neq 1$. Hence $\bar{\Gamma}$ is not metaabelian. QED

Claim 1. If conditions (i)-(viii) and (c1)-(c3) hold then the element b acts freely.

Proof. Assuming the opposite let $Fix(b) \neq \emptyset$. By condition (c1), we have $0 \notin Fix(b)$. Then let $x \in Fix(b)$ such that $|x|$ is minimal. Without loss of generality, let us assume that $x > 0$. Then by condition (iv), $x \in Fix(a)$. Then by condition (c3), $x \notin Fix(t)$ thus there exists $y > 0$ such that $y \neq x$ and $t(y) = x$.

Let $x_1 \in Fix(a) \cap (0, \infty)$ such that x_1 is minimal. Then, by condition (iv), $x_1 \in Fix(t)$. Then, by condition (c3), we have $0 < x_1 < x$. Moreover, since $x_1 \in Fix(t)$, we have $y > x_1$.

Without loss of generality, we may assume that $y < x$. Let $d = t^{-1}bt$. Notice that $g(0) < x_1$ for all $g \in G$. Then by (c2) there exists $b_n = t^{-n}bt^n \in S, n > 0$ such that $x_1 < b_n(0) < y$. Then $[b_n^{-1}, d] \neq 1$. Indeed, first, notice that $Fix(b_n) \cap [0, x] = Fix(b) \cap [0, x]$. Furthermore, we have $d(y) = t^{-1}bt(y) = y$. Hence $b_n^{-1}d(y) = b_n^{-1}(y)$. But $0 < b_n^{-1}(y) < y$. Hence $db_n^{-1}(y) \neq b_n^{-1}(y)$.

The inequality $[b_n^{-1}, t^{-1}bt] \neq 1$ contradicts (v). $\square$

Construction of Γ: Let us now construct Γ with properties (i)-(vii).

We consider the rings

$$A = \mathbb{Z}, B = \mathbb{Z}[\frac{1}{2}], D = \mathbb{Z}[\frac{1}{2}, \sqrt{2}, \frac{1}{\sqrt{2}}, \sqrt[4]{2}, \frac{1}{\sqrt[4]{2}}, \sqrt[8]{2}, \frac{1}{\sqrt[8]{2}}, \ldots]$$

We will identify t, a, b with the identity elements of the rings A, B, D respectively.

Let $\Omega = \oplus_{n\in\mathbb{Z}} H_n$ where H_n is isomorphic to the additive group of the ring D. An element of Ω can be represented by a vector $\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)$ where all but finitely many coordinates are zero. We choose a special two sided sequence $\mathbf{c} = (\ldots, c_{-1}, c_0, c_1, \ldots)$ where $c_0 = 2$, and $c_{n+1} = c_n^2$ for all $n \in \mathbb{Z}$. [So $c_n = 2^{2^n}, \forall n \in \mathbb{Z}$]

The Baumslag-Solitar group $BS(1,2) = \langle t, a \mid tat^{-1} = a^2 \rangle$ acts on Ω as follows:

for all $\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \Omega$,

$t(\mathbf{x}) = \mathbf{y}$ where $\mathbf{y} = (\ldots, y_{-1}, y_0, y_1, \ldots), y_n = x_{n-1}, \forall n \in \mathbb{Z}$. (so t acts by a shift);

$a(\mathbf{x}) = \mathbf{y}$ where $\mathbf{y} = (\ldots, y_{-1}, y_0, y_1, \ldots), y_n = c_n x_n, \forall n \in \mathbb{Z}$.

Let Γ be the semidirect product $BS(1,2) \ltimes \Omega$ with respect to the described action.

The group Γ is generated by three elements t, a, b where $b = (\ldots, b_{-1}, b_0, b_1, \ldots), b_0 = 1, b_n = 0, \forall n \neq 0$. (thus we identify t, a with the identity elements of the rings A, B respectively; and we identify b with the identity element of the isomorphic copy H_0 of the ring D).

By construction, Γ satisfies conditions (i), (iii), (iv), and (v). To discuss conditions (ii), (vi) and (vii), let us recall a basic fact about left-orderable groups.

Lemma. Let a group G_1 acts on a group G_2 by automorphisms. Let $\prec_1, \prec_2$ be left orders on G_1, G_2 respectively, and assume that the action of G_1 on G_2 preserves the left order [i.e. if $g \in G_1, x_1, x_2 \in G_2, x_1 \prec_2 x_2$ then $g(x_1) \prec_2 g(x_2)$].

Then there exists a left order $<$ in $G_1 \ltimes G_2$ which satisfies the following conditions:

1) if $g_1, f_1 \in G_1, g_1 \prec_1 f_1$ then $(g_1, 1) < (f_1, 1)$;

2) if $g_2, f_2 \in G_2, g_2 \prec_2 f_2$ then $(1, g_2) < (1, f_2)$;

3) if $g_1 \in G_1 \backslash \{1\}, g_2 \in G_2 \backslash \{1\}, 1 \prec_2 g_2$, then $(g_1, 1) < (1, g_2)$.

Proof. We define the left order on $G_1 \ltimes G_2$ as follows: given $(g_1, f_1), (g_2, f_2) \in G_1 \ltimes G_2$ we define $(g_1, f_1) < (g_2, f_2)$ iff either $f_1 \prec_2 f_2$ or $f_1 = f_2, g_1 \prec_1 g_2$. Then the claim is a direct check. $\square$

The left order $<$ on the semidirect product $G_1 \ltimes G_2$ constructed in the proof of the lemma will be called the *extension of* $\prec_1$ *and* $\prec_2$.

Let us now introduce left orders $\prec_1, \prec_2$ on the additive subgroups of the rings A, B respectively. Notice that the additive groups A, B are subgroups of $\mathbb{R}$, and we define $\prec_1, \prec_2$ to be simply the restrictions of the natural order of $\mathbb{R}$.

Now we introduce an order $\prec_3$ on Ω. Let $\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, \ldots), \mathbf{y} = (\ldots, y_{-1}, y_0, y_1, \ldots) \in \Omega$. We say $\mathbf{x} \prec_3 \mathbf{y}$ iff either $\sum_{i \in \mathbb{Z}} x_i < \sum_{i \in \mathbb{Z}} y_i$ or $\sum_{i \in \mathbb{Z}} x_i = \sum_{i \in \mathbb{Z}} y_i, \min\{k \mid x_k < y_k\} < \min\{k \mid y_k < x_k\}$.

Notice that $BS(1,2) = \langle t, a \mid tat^{-1} = a^2 \rangle$ is isomorphic to $A \ltimes B$ (where we take the standard action of A on B, i.e. by multiplication by 2). Notice that the action of A on B preserves the left order $\prec_2$. Then, let $\prec_4$ be the extension of $\prec_1$ and $\prec_2$. Having the left order $\prec_4$ on $BS(1,2)$, we define the left order $<$ on $\Gamma = BS(1,2) \ltimes \Omega$ to be the extension of $\prec_4$ and $\prec_3$ [again, notice that the action of $BS(1,2)$ on Ω preserves the left order $\prec_3$]. The group $\Gamma = BS(1,2) \ltimes \Omega$ with the left order $<$ satisfies condition (ii), (vi), (vii) and (viii). $\square$

Remark 3. A.Navas pointed out a non-trivial fact that the construction presented in this paper has no realization in C^2-regularity. It is indeed interesting (as suggested by A.Navas) if the methods of [Be] can still be useful in smooth category.

R e f e r e n c e s

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